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On a conjecture about the kth lower multiexponent[☆]

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ABSTRACT

In this paper, the conjecture about the kth lower multiexponent $f(n, k)$ proposed by R.A. Brualdi and B. Liu is proved to be true for the following cases: (1) $k = n - i$, where $i = 2, 3, 4, 5$; (2) small n, where $n \leq 8$; (3) the class of primitive micro-symmetric digraphs of order n.

1. Introduction

A digraph D is called primitive if and only if D is strongly connected and the greatest common divisor $g.c.d.(r_1, \ldots, r_s) = 1$, where $\{r_1, \ldots, r_s\}$ is the set of distinct lengths of the directed cycles in D [1]. Let D be a primitive digraph with vertex set $V = \{1, \ldots, n\}$, and let $X \subseteq V$. The exponent of the set X is the least integer m such that for each vertex i of D there exists a walk from at least one vertex in X to i of length m, denoted by $\exp_D(X)$ [2].

In 1990, Brualdi and Liu [2] introduced the kth lower multiexponent of a primitive digraph D with n vertices as follows: for $1 \leq k \leq n$,

$$f(D, k) := \min\{\exp_D(X) \mid X \subseteq V \text{ and } |X| = k\},$$

and

$$f(n, k) := \max_D\{f(D, k)\},$$

where the maximum is taken over all primitive digraphs of order n.

In [2], the authors proved that

$$f(n, k) = \begin{cases} n^2 - 3n + 3, & k = 1, \\ 1, & k = n - 1, \\ 0, & k = n, \end{cases}$$

and they proposed the following conjecture about $f(n, k)$.

Conjecture 1.1 (*[2]*). *For any integers n, k with $2 \leq k \leq n - 2$,*

$$f(n, k) = 1 + (2n - k - 2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 \cdot k.$$

It can be seen that the above equality is also true for $k = 1, n - 1$.

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The kth lower multiexponent of a primitive digraph has been studied by many. In particular, in [3–7], Conjecture 1.1 has been verified for several classes of digraphs, including primitive digraphs with a directed cycle whose length is divisible k, primitive simple graphs, primitive tournaments, primitive symmetric digraphs, etc.

In this paper, we prove that Conjecture 1.1 holds for the following cases:

(1) $k = n-2, n-3, n-4, n-5$;
(2) small n, where $n \le 8$;
(3) the class of primitive micro-symmetric digraphs of order n.

2. Preliminaries

Let D_n be a primitive digraph with vertices $1, 2, \ldots, n$ and arcs $1 \to n \to n-1 \to \cdots \to 2 \to 1$ and $1 \to n-1$, where $n \ge 2$. It is well known that D_n is called the Wielandt digraph [1]. The kth lower multiexponent of the Wielandt digraph had been investigated in [2,1].

Lemma 2.1 ([2,1]). *Let n, k be positive integers with $1 \le k \le n-1$. Then*

$$f(D_n, k) = 1 + (2n-k-2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 \cdot k.$$

By Lemma 2.1, D_n is an extremal digraph reaching the bound given by $f(n, k)$.

Let $\overline{D_n}$ be a primitive digraph obtained from D_n by adding an arc $2 \to n$, where $n \ge 4$. Note that D_n is a subdigraph of $\overline{D_n}$, then for $1 \le k \le n-1$,

$$f\left(\overline{D_n}, k\right) \le f(D_n, k).$$

For convenience, by an l-dicycle we mean a directed cycle of length l. The following lemmas give some upper bounds of $f(D, k)$ for different cases, which are useful in this paper.

Lemma 2.2 ([2]). *Let D be a primitive digraph of order n. Suppose that D has a s-dicycle. Then for any integer k with $s \le k \le n$,*

$$f(D, k) \le n - k.$$

Lemma 2.3 ([3]). *Let D be a primitive digraph with n vertices. If there is a s-dicycle intersecting with a $(s+1)$-dicycle (or with a $(s+2)$-dicycle, where s is odd) in D, then for $k < s$, we have*

$$f(D, k) \le 1 + (2n-k-2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 \cdot k.$$

Lemma 2.4 ([1,3]). *Let D be a primitive digraph with n vertices which contains a s-dicycle, where $1 \le s \le n-1$. Then for $k | s$,*

$$f(D, k) \le 1 + (2n-k-2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 \cdot k.$$

Let $X \subseteq V(D)$ and let $R_t(X)$ be the set of vertices in D, which can be reached by a walk of length t from some vertex in X, where t is a nonnegative integer.

Lemma 2.5. *Let $H_n^{(1)}$ be a primitive digraph with vertices $1, 2, \ldots, n$ and arcs $1 \to n \to n-1 \to \cdots \to 2 \to 1$ and $1 \to n-3$. Then*

$$f\left(H_n^{(1)}, n-4\right) \le 7 \quad \text{for } n \ge 7$$

and

$$f\left(H_n^{(1)}, n-5\right) \le 8 \quad \text{for } n \ge 8.$$

Proof. Let $X_1 = \{1, 2, \ldots, n\} - \{2, 4, n-2, n\}$ be a set of $(n-4)$ vertices, where $n \ge 7$. It is not difficult to verify that

$$R_0(X_1) = X_1, \qquad R_1(X_1) = \{1, 2, \ldots, n\} - \{1, 3, n-1\},$$
$$R_2(X_1) = \{1, 2, \ldots, n\} - \{2, n-2, n\}, \ldots,$$
$$R_6(X_1) = \{1, 2, \ldots, n\} - \{n-2\}, \qquad R_7(X_1) = \{1, 2, \ldots, n\}.$$

Then by the definition of kth lower multiexponent, we have

$$f\left(H_n^{(1)}, n-4\right) \le 7 \quad \text{for } n \ge 7.$$

Now let $X_2 = \{1, 2, \dots, n\} - \{4, 5, n-2, n-1, n\}$ be a set of $n-5$ vertices, where $n \ge 8$. Observe that

$$R_0(X_2) = X_2, \qquad R_1(X_2) = \{1, 2, \dots, n\} - \{3, 4, n-2, n-1\},$$
$$R_2(X_2) = \{1, 2, \dots, n\} - \{2, 3, n-2\}, \dots,$$
$$R_7(X_2) = \{1, \dots, n\} - \{n-2\}, \qquad R_8(X_2) = \{1, 2, \dots, n\}.$$

By the definition of kth lower multiexponent, it follows

$$f\left(H_n^{(1)}, n-5\right) \le 8,$$

where $n \ge 8$. This completes the proof of Lemma 2.5. □

Lemma 2.6. *Let $H_n^{(2)}$ be a primitive digraph with vertices $1, 2, \dots, n$ and arcs $1 \to n \to n-1 \to \cdots \to 2 \to 1$ and $1 \to n-4$, where $n \ge 9$. Then*

$$f\left(H_n^{(2)}, n-5\right) \le 9.$$

Proof. Let $X = \{1, 2, \dots, n\} - \{3, 5, n-3, n-2, n\}$ be a set of $n-5$ vertices, where $n \ge 9$. Observe that

$$R_0(X) = X, \qquad R_1(X) = \{1, 2, \dots, n\} - \{2, 4, n-3, n-1\},$$
$$R_2(X) = \{1, \dots, n\} - \{1, 3, n-2\}, \dots,$$
$$R_8(X) = \{1, 2, \dots, n\} - \{n-3\}, \qquad R_9(X) = \{1, 2, \dots, n\}.$$

Then by the definition of kth lower multiexponent,

$$f\left(H_n^{(2)}, n-5\right) \le 9,$$

where $n \ge 9$. This completes the proof. □

Lemma 2.7. *Let $H_n^{(3)}$ be a primitive digraph with vertices $1, 2, \dots, n$ and arcs $1 \to n-1 \to n-2 \to \cdots \to 2 \to 1$ and $1 \to n \to n-5$, where $n \ge 8$. Then*

$$f\left(H_n^{(3)}, n-5\right) \le 8.$$

Proof. Let $X = \{1, 2, \dots, n\} - \{2, 4, n-3, n-1, n\}$ be a set of $n-5$ vertices, where $n \ge 8$. Notice that

$$R_0(X) = X, \qquad R_1(X) = \{1, 2, \dots, n\} - \{1, 3, n-4, n-2\},$$
$$R_2(X) = \{1, \dots, n\} - \{2, n-3, n-1, n\}, \dots,$$
$$R_7(X) = \{1, 2, \dots, n\} - \{n-4\}, \qquad R_8(X) = \{1, 2, \dots, n\}.$$

It follows from the definition of kth lower multiexponent that

$$f\left(H_n^{(3)}, n-5\right) \le 8,$$

where $n \ge 8$. Therefore, we obtain the result as desired. □

Lemma 2.8. *Let $H_n^{(4)}$ be a primitive digraph obtained from $H_{n-1}^{(1)}$ by adding a vertex n and inserting some arcs, where $n \ge 8$. Then*

$$f\left(H_n^{(4)}, n-5\right) \le 8.$$

Proof. Let $X = \{1, 2, \dots, n\} - \{2, 4, n-3, n-1, n\}$ be a set of $n-5$ vertices, where $n \ge 8$. It is not difficult to check that

$$R_0(X) = X, \qquad R_1(X) \supseteq \{1, 2, \dots, n-1\} - \{1, 3, n-2\},$$
$$R_2(X) \supseteq \{1, \dots, n-1\} - \{2, n-3, n-1\}, \dots,$$
$$R_6(X) \supseteq \{1, \dots, n-1\} - \{n-3\}, \qquad R_7(X) \supseteq \{1, 2, \dots, n-1\}.$$

Note that $H_n^{(4)}$ is primitive, and there exists an arc from at least one vertex in $\{1, 2, \dots, n-1\}$ to vertex i for each $i = 1, 2, \dots, n$. Hence

$$R_8(X) = \{1, 2, \dots, n-1, n\}.$$

Combining this with the definition of kth lower multiexponent,

$$f\left(H_n^{(4)}, n-5\right) \le 8,$$

where $n \ge 8$. The proof is finished. □

Let $\widetilde{D_n}$ be a primitive digraph with vertices $1, 2, \ldots, n$ and arcs $1 \to n \to n-1 \to \cdots \to 2 \to 1$ and $1 \to 3$, where $n \geq 5$. Now the numbers $f(7, 2)$ and $f(8, 2)$ are given as follows.

Lemma 2.9. $f(7, 2) = 13$.

Proof. On one hand, by Lemma 2.1,

$$f(D_7, 2) = 1 + (14 - 2 - 2)\left\lfloor \frac{6}{2} \right\rfloor - \left\lfloor \frac{6}{2} \right\rfloor^2 \cdot 2 = 13.$$

Now let D be a primitive digraph of order 7, and let s be the girth of D.

Case 1. $s = 6$. We conclude that D is isomorphic to D_7 or $\overline{D_7}$, and then

$$f\left(\overline{D_7}, 2\right) \leq f(D_7, 2) = 13.$$

Case 2. $s = 5$. Since D is primitive, there exists a 6-dicycle (or 7-dicycle). Note that the 5-dicycle and 6-dicycle (resp. 5-dicycle and 7-dicycle) intersect, then by Lemma 2.3, we have

$$f(D, 2) \leq 1 + (14 - 2 - 2)\left\lfloor \frac{6}{2} \right\rfloor - \left\lfloor \frac{6}{2} \right\rfloor^2 \cdot 2 = 13. \tag{1}$$

Case 3. $s = 4$. Note that there is a 4-dicycle in D, and $2|4$. By Lemma 2.4, it is not difficult to see that Inequality (1) holds.

Case 4. $s = 3$.

If D contains a 4-dicycle (or 6-dicycle), since $2|4$ and $2|6$, it follows from Lemma 2.4 that Inequality (1) holds.

If D contains a 5-dicycle, note that the 3-dicycle and 5-dicycle intersect, then by Lemma 2.3, Inequality (1) holds.

Otherwise, there exists a 7-dicycle, but no directed cycles of lengths 4, 5 and 6. It follows that D is isomorphic to a primitive digraph D^* obtained from the digraph $\widetilde{D_7}$ by inserting some edges. Let $X = \{1, 3\}$ be a set of two vertices of $\widetilde{D_7}$. Considering the digraph $\widetilde{D_7}$, we observe that

$$R_0(X) = X, \qquad R_1(X) = \{2, 3, 7\}, \qquad R_2(X) = \{1, 2, 6\},$$
$$R_3(X) = \{1, 3, 5, 7\}, \ldots,$$
$$R_8(X) = \{1, 2, 3, 5, 6, 7\}, \qquad R_9(X) = \{1, 2, \ldots, 7\}.$$

Then by the definition of kth lower multiexponent, and notice that $\widetilde{D_7}$ is a subdigraph of D^*, we obtain that

$$f\left(D^*, 2\right) \leq f\left(\widetilde{D_7}, 2\right) \leq 9 < 13.$$

Case 5. $s \leq 2$. It follows from Lemma 2.2 that

$$f(D, 2) \leq 7 - 2 = 5 < 13.$$

From Cases 1 to 5, we conclude that $f(7, 2) = 13$. □

Lemma 2.10. $f(8, 2) = 19$.

Proof. On one hand, it follows from Lemma 2.1 that

$$f(D_8, 2) = 1 + (16 - 2 - 2)\left\lfloor \frac{7}{2} \right\rfloor - \left\lfloor \frac{7}{2} \right\rfloor^2 \cdot 2 = 19.$$

On the other hand, let D be a primitive digraph of order 8, and let s denote the girth of D. We need to show that $f(D, 2) \leq 19$.

Case 1. $s = 7$. Then D is isomorphic to D_8 or $\overline{D_8}$, and it follows

$$f\left(\overline{D_8}, 2\right) \leq f(D_8, 2) = 19.$$

Case 2. $s = 6$ or 4. Note that there is a 6-dicycle (or 4-dicycle) in D, $2|6$ and $2|4$, it follows from Lemma 2.4 that

$$f(D, 2) \leq 19. \tag{2}$$

Case 3. $s = 5$.

If D contains a 6-dicycle, since $2|6$ and by Lemma 2.4, Inequality (2) holds.

If D contains a 7-dicycle, since the 5-dicycle and 7-dicycle intersect, then by Lemma 2.3, Inequality (2) holds.

Otherwise, there exists an 8-dicycle, but no directed cycles of lengths 6 and 7. Thus D is isomorphic to a primitive digraph D^* obtained from $H_8^{(1)}$ by inserting some edges. Let $X = \{1, 5\}$ be a set of two vertices of $H_8^{(1)}$. It is easy to see that for the digraph $H_8^{(1)}$,

$$R_0(X) = X, \qquad R_1(X) = \{4, 5, 8\}, \qquad R_2(X) = \{3, 4, 7\}, \ldots,$$
$$R_{14}(X) = \{1, 2, \ldots, 8\} - \{6\}, \qquad R_{15}(X) = \{1, 2, \ldots, 8\}.$$

Then by the definition of kth lower multiexponent, we obtain that

$$f\left(D^{*},2\right)\leq f\left(H_{8}^{(1)},2\right)\leq 15<19.$$

Case 4. $s=3$.

Subcase 4.1. D contains a 4-dicycle or 6-dicycle. Since $2|4$ and $2|6$, it follows from Lemma 2.4 that Inequality (2) holds.

Subcase 4.2. D contains a 7-dicycle and an 8-dicycle. Clearly, these two directed cycles intersect.

Subcase 4.3. D contains a 5-dicycle, which is intersected by a 3-dicycle.

For Subcases 4.2 and 4.3, by Lemma 2.3, Inequality (2) holds.

Subcase 4.4. There exists an 8-dicycle, but no directed cycles of lengths 4, 6 and 7. It follows that D is isomorphic to a primitive digraph D^{**} obtained from the digraph $\widetilde{D_8}$ by inserting some edges. Let $X=\{1,2\}$ be a set of two vertices of $\widetilde{D_8}$. Observe that in $\widetilde{D_8}$,

$$R_0(X)=X,\qquad R_1(X)=\{1,3,8\},\qquad R_2(X)=\{2,3,7,8\},\ldots,$$
$$R_{10}(X)=\{1,2,\ldots,8\}-\{4\},\qquad R_{11}(X)=\{1,2,\ldots,8\}.$$

Then by the definition of kth lower multiexponent, and notice that $\widetilde{D_8}$ is a subdigraph of D^{**}, we have

$$f\left(D^{**},2\right)\leq f\left(\widetilde{D_8},2\right)\leq 11<19.$$

Subcase 4.5. There exists a 7-dicycle, but no directed cycles of lengths 4, 6 and 8. Besides, if there is a 5-dicycle, then the 5-dicycle does not intersect with any 3-dicycles.

Therefore, we conclude that D is isomorphic to: (I) a primitive digraph $D^{(1)}$ obtained from the digraph $\widetilde{D_7}$ by adding a vertex n and inserting some edges; (II) a primitive digraph $D^{(2)}$ obtained from the digraph Q_8 by inserting some edges, where Q_8 is a primitive digraph with vertices $1,2,\ldots,8$ and arcs $1\to 7\to 6\to\cdots\to 2\to 1$ and $1\to 8\to 2$.

For (I), from the forgoing proof, we know that

$$f\left(\widetilde{D_7},2\right)\leq 9.$$

Note that $D^{(1)}$ is primitive, and there is an arc from some vertex in $\{1,2,\ldots,7\}$ to vertex i for each $i=1,2,\ldots,8$. Therefore,

$$f\left(D^{(1)},2\right)\leq f\left(\widetilde{D_7},2\right)+1\leq 10<19.$$

For (II), let $X=\{1,2\}$ be a set of two vertices of Q_8. Observe that

$$R_0(X)=X,\qquad R_1(X)=\{1,7,8\},\qquad R_2(X)=\{2,6,7,8\},\ldots,$$
$$R_{10}(X)=\{1,2,\ldots,8\}-\{3\},\qquad R_{11}(X)=\{1,2,\ldots,8\}.$$

By the definition of kth lower multiexponent, we have

$$f\left(D^{(2)},2\right)\leq f\left(Q_8,2\right)\leq 11<19.$$

Case 5. $s\leq 2$. Then by Lemma 2.2, we have

$$f(D,2)\leq 8-2=6<19.$$

Consequently, we conclude that $f(8,2)=19$. □

3. Main results

To begin with, we show that Conjecture 1.1 holds for $n-2\leq k\leq n-5$.

Theorem 3.1. *Let n be an integer with $n\geq 4$. Then*

$$f(n,n-2)=1+[2n-(n-2)-2]\left\lfloor\frac{n-1}{n-2}\right\rfloor-\left\lfloor\frac{n-1}{n-2}\right\rfloor^{2}\cdot(n-2)=3.$$

Proof. Let D be a primitive digraph of order n with $n\geq 4$, and let s denote the girth of D. It is known that $f(D_n,n-2)=3$, where $n\geq 4$. It will suffice to show that $f(D,n-2)\leq 3$.

Case 1. $s\leq n-2$. Then by Lemma 2.2,

$$f(D,n-2)\leq n-(n-2)=2<3.$$

Case 2. $s=n-1$. Then D is isomorphic to D_n or $\overline{D_n}$. Hence

$$f\left(\overline{D_n},n-2\right)\leq f\left(D_n,n-2\right)=3.$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let n be an integer with $n \geq 5$. Then*

$$f(n, n-3) = 1 + [2n - (n-3) - 2]\left\lfloor \frac{n-1}{n-3} \right\rfloor - \left\lfloor \frac{n-1}{n-3} \right\rfloor^2 \cdot (n-3) = 5.$$

Proof. On one hand, it follows from Lemma 2.1 that

$$f(D_n, n-3) = 5 \quad \text{for } n \geq 5.$$

On the other hand, let D be a primitive digraph of order n, where $n \geq 5$. Let s denote the girth of D. We need to show that $f(D, n-3) \leq 5$.

Case 1. $s \leq n-3$. By Lemma 2.2, we have

$$f(D, n-3) \leq n - (n-3) = 3 < 5.$$

Case 2. $s = n-2$.

Subcase 2.1. There exists an $(n-1)$-dicycle in D. Since $n \geq 5$, there is an $(n-2)$-dicycle intersecting with an $(n-1)$-dicycle in D. By Lemma 2.3,

$$f(D, n-3) \leq 5.$$

Subcase 2.2. D contains no $(n-1)$-dicycles. Clearly, there is an n-dicycle in D, and n is odd. Moreover, the $(n-2)$-dicycle and n-dicycle intersect since $n \geq 5$. Then by Lemma 2.3,

$$f(D, n-3) \leq 5.$$

Case 3. $s = n-1$. Then D is isomorphic to D_n or $\overline{D_n}$. Hence

$$f\left(\overline{D_n}, n-3\right) \leq f(D_n, n-3) = 5.$$

Combining the above cases, we obtain the desired result. □

Theorem 3.3. *Let n be an integer with $n \geq 6$. Then*

$$f(n, n-4) = 1 + [2n - (n-4) - 2]\left\lfloor \frac{n-1}{n-4} \right\rfloor - \left\lfloor \frac{n-1}{n-4} \right\rfloor^2 \cdot (n-4) = \begin{cases} 9, & n = 6, \\ 7, & n \geq 7. \end{cases}$$

Proof. By Lemma 2.1, we know that

$$f(D_n, n-4) = \begin{cases} 9, & n = 6, \\ 7, & n \geq 7. \end{cases}$$

Let D be a primitive digraph of order n, and let s denote the girth of D, where $n \geq 6$. Now we show that $f(D, n-4) \leq \begin{cases} 9, & n = 6, \\ 7, & n \geq 7. \end{cases}$

Case 1. $s \leq n-4$. Then by Lemma 2.2,

$$f(D, n-4) \leq n - (n-4) = 4 < 7.$$

Case 2. $s = n-3$.

Subcase 2.1. There is either an $(n-2)$-dicycle, or an $(n-1)$-dicycle and an n-dicycle. Then D contains two intersecting directed cycles of lengths $n-3$ and $n-2$ (or $n-1$ and n). By Lemma 2.3,

$$f(D, n-4) \leq \begin{cases} 9, & n = 6, \\ 7, & n \geq 7. \end{cases} \tag{3}$$

Subcase 2.2. D contains no $(n-2)$-dicycles and $(n-1)$-dicycles, then D contains an n-dicycle. Hence $g.c.d.(n, n-3) = 1$, and then D is isomorphic to a primitive digraph D^* obtained from $H_n^{(1)}$ by inserting some edges, where $n \geq 7$. Since $H_n^{(1)}$ is a subdigraph of D^*, by Lemma 2.5,

$$f\left(D^*, n-4\right) \leq f\left(H_n^{(1)}, n-4\right) \leq 7.$$

Subcase 2.3. D contains no $(n-2)$-dicycles and n-dicycles, then there is an $(n-1)$-dicycle in D, and n is an even number. Since $n \geq 6$, we conclude that the $(n-3)$-dicycle and $(n-1)$-dicycle intersect. Moreover, since $n-1$ is odd, by Lemma 2.3, Inequality (3) holds.

Case 3. $s = n-2$. Similarly as the discussion of Theorem 3.2 Case 2, it follows from Lemma 2.3 that Inequality (3) holds.

Case 4. $s = n - 1$. It follows that D is isomorphic to D_n or $\overline{D_n}$. Then

$$f\left(\overline{D_n}, n-4\right) \le f\left(D_n, n-4\right) = \begin{cases} 9, & n = 6, \\ 7, & n \ge 7. \end{cases}$$

Combining the above cases, there follows the result as desired. □

Theorem 3.4. *Let n be an integer with $n \ge 8$. Then*

$$f(n, n-5) = 1 + [2n - (n-5) - 2]\left\lfloor \frac{n-1}{n-5} \right\rfloor - \left\lfloor \frac{n-1}{n-5} \right\rfloor^2 \cdot (n-5) = \begin{cases} 11, & n = 8, \\ 9, & n \ge 9. \end{cases}$$

Proof. On one hand, by Lemma 2.1, we have

$$f\left(D_n, n-5\right) = \begin{cases} 11, & n = 8, \\ 9, & n \ge 9. \end{cases}$$

On the other hand, let D be a primitive digraph of order n, and let s be the girth of D, where $n \ge 8$. It will suffice to show that $f(D, n-5) \le \begin{cases} 11, & n = 8, \\ 9, & n \ge 9. \end{cases}$

Case 1. $s \le n - 5$. Then by Lemma 2.2,

$$f(D, n-5) \le n - (n-5) = 5 < 9.$$

Case 2. $s = n - 4$.

Subcase 2.1. There exists an $(n-3)$-dicycle in D. Since $n \ge 8$, the $(n-4)$-dicycle and $(n-3)$-dicycle intersect.

Subcase 2.2. There exist an $(n-1)$-dicycles and an n-dicycles. Obviously, these two directed cycles intersect.

Subcase 2.3. There exists an $(n-2)$-dicycle, and n is odd. Clearly, the $(n-4)$-dicycle and $(n-2)$-dicycle intersect.

For Subcases 2.1–2.3, it follows from Lemma 2.3 that

$$f(D, n-5) \le \begin{cases} 11, & n = 8, \\ 9, & n \ge 9. \end{cases} \tag{4}$$

Subcase 2.4. D contains no $(n-3)$-dicycles and $(n-1)$-dicycles, while contains an n-dicycle. If n is even and there is an $(n-2)$-dicycle in D, then the numbers $n, n-2, n-4$ are all even, which is a contradiction. Hence we can suppose that there exists no $(n-2)$-dicycles in D for this subcase.

Consequently, $g.c.d.(n, n-4) = 1$, and D is isomorphic to a primitive digraph D^* obtained from the digraph $H_n^{(2)}$ by inserting some edges, where $n \ge 9$. Since $H_n^{(2)}$ is a subdigraph of D^*, it follows from Lemma 2.6 that

$$f\left(D^*, n-5\right) \le f\left(H_n^{(2)}, n-5\right) \le 9.$$

Subcase 2.5. D contains no $(n-3)$-dicycles and n-dicycles, while contains an $(n-1)$-dicycle in D.

If there is an $(n-2)$-dicycle in D, then the $(n-2)$-dicycle and $(n-1)$-dicycle intersect. By Lemma 2.3, Inequality (4) holds.

If there is no $(n-2)$-dicycles in D, then $g.c.d.(n-1, n-4) = 1$, and D is isomorphic to a primitive digraph D^{**} obtained from the digraph $H_n^{(3)}$ or $H_n^{(4)}$ by inserting some edges, where $n \ge 8$. Since $H_n^{(3)}$ (resp. $H_n^{(4)}$) is a subdigraph of D^{**}, by Lemmas 2.7 and 2.8, we have

$$f\left(D^{**}, n-5\right) \le \begin{cases} f\left(H_n^{(3)}, n-5\right) \\ f\left(H_n^{(4)}, n-5\right) \end{cases} \le 8 < 9.$$

Case 3. $s = n - 3$. Analogously as the discussion of Theorem 3.3 Case 2, it follows from Lemmas 2.3 and 2.5 that

$$f(D, n-5) \le \begin{cases} 11, & n = 8, \\ 9, & n \ge 9. \end{cases}$$

or

$$f(D, n-5) \le f\left(H_n^{(1)}, n-5\right) \le 8 < 9 \quad \text{for } n \ge 8.$$

Case 4. $s = n - 2$. Similarly as the discussion of Theorem 3.2 Case 2, it follows from Lemma 2.3 that Inequality (4) holds.

Case 5. $s = n - 1$. Then D is isomorphic to D_n or $\overline{D_n}$. Therefore,

$$f\left(\overline{D_n}, n-5\right) \le f\left(D_n, n-5\right) = \begin{cases} 11, & n = 8, \\ 9, & n \ge 9. \end{cases}$$

Combining the above cases, the proof is finished. □

Note that $2 \leq k \leq n-2$ in Conjecture 1.1, then $n \geq 4$. We show that Conjecture 1.1 holds for $4 \leq n \leq 8$ in the following theorem.

Theorem 3.5. *Let n, k be integers with $4 \leq n \leq 8$ and $2 \leq k \leq n-2$. Then*

$$f(n,k) = 1 + (2n-k-2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 \cdot k.$$

Proof. (1) $n = 4$. Then $k = 2$. By Theorem 3.1, it is obvious that

$$f(4,2) = 3.$$

(2) $n = 5$. Then $k = 2, 3$. It follows from Theorems 3.1 and 3.2 that

$$f(5,3) = 3, \quad \text{and} \quad f(5,2) = 5.$$

(3) $n = 6$. Then $k = 2, 3, 4$. By Theorems 3.1–3.3, we have

$$f(6,4) = 3, \qquad f(6,3) = 5, \quad \text{and} \quad f(6,2) = 9.$$

(4) $n = 7$. Then $k = 2, 3, 4, 5$. From Theorems 3.1–3.3 and Lemma 2.9,

$$f(7,5) = 3, \qquad f(7,4) = 5, \qquad f(7,3) = 7, \quad \text{and} \quad f(7,2) = 13.$$

(5) $n = 8$. Then $k = 2, 3, 4, 5, 6$. By Theorems 3.1–3.4 and Lemma 2.10, it is not difficult to obtain that

$$f(8,6) = 3, \qquad f(8,5) = 5, \qquad f(8,4) = 7, \qquad f(8,3) = 11 \quad \text{and} \quad f(8,2) = 19.$$

All in all, for $4 \leq n \leq 8$ and $2 \leq k \leq n-2$, we conclude that

$$f(n,k) = 1 + (2n-k-2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 \cdot k.$$

This completes the proof of Theorem 3.5. □

A digraph D is called a micro-symmetric digraph if there is a pair i, j with $i \neq j$, such that (i, j) and (j, i) are both arcs. Let n be an integer with $n \geq 2$. Let MS_n denote the set of all primitive micro-symmetric digraphs of order n ([8,9]). Now we show that Conjecture 1.1 holds for $D \in MS_n$.

Theorem 3.6. *Let $D \in MS_n$, where $n \geq 4$. Then for $2 \leq k \leq n-2$,*

$$f(D,k) \leq n-k \leq 1 + (2n-k-2)\left\lfloor \frac{n-1}{k} \right\rfloor - k \cdot \left\lfloor \frac{n-1}{k} \right\rfloor^2.$$

The first inequality can be attained if D is isomorphic to the digraph S_n, where S_n $(n \geq 4)$ is a primitive micro-symmetric digraph with vertices $1, 2, \ldots, n$ and arcs $1 \to n \to n-1 \to \cdots \to 2 \to 1$ and $1 \to 2$.

Proof. Since $D \in MS_n$, D contains a 2-dicycle. For $2 \leq k \leq n-2$, it follows from Lemma 2.2 that

$$f(D,k) \leq n-k.$$

Moreover, since

$$1 \leq \left\lfloor \frac{n-1}{k} \right\rfloor \leq \frac{n-1}{k},$$

it is obvious that

$$\begin{aligned} 1 + (2n-k-2)\left\lfloor \frac{n-1}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor^2 \cdot k &= 1 + \left\lfloor \frac{n-1}{k} \right\rfloor \cdot \left[(2n-k-2) - \left\lfloor \frac{n-1}{k} \right\rfloor \cdot k\right] \\ &\geq 1 + 1 \cdot \left[(2n-k-2) - \frac{n-1}{k} \cdot k\right] \\ &= n-k \\ &\geq f(D,k). \end{aligned}$$

Now we consider the primitive micro-symmetric digraph S_n. Let $R_t(X)$ denote the set of vertices of S_n, which can be reached by a walk of length t from some vertex of X, where $X \subseteq V(S_n)$, and t is nonnegative. Observe that

$$R_{n-k-1}(\{1,2\}) = \{1, 2, n, n-1, \dots, k+2\},$$
$$R_{n-k-1}(\{i\}) \subseteq R_{n-k-1}(\{1,2\}), \quad \text{where } i = 3, 4, \dots, n-k+1,$$
$$R_{n-k-1}(\{j\}) = \{j+k+1-n\}, \quad \text{where } j = n, n-1, \dots, n-k+2.$$

Then for any set Y of k vertices, we have

$$|R_{n-k-1}(Y)| \le n-1 < n.$$

Combining this with the fact that $S_n \in MS_n$, it follows that

$$n-k-1 < f(S_n, k) \le n-k.$$

Hence $f(S_n, k) = n-k$ for $2 \le k \le n-2$. This completes the proof. $\square$

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